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CHOW POLYTOPES

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The main results presented in this talk were obtained in a joint work with M.M. Kapranov and B. Sturmfels (Chow polytopes and general resultants, Technical report '91-2, MSI, Cornell University, January 1991). Let X be a projective variety of dimension $k-1$ embedded into the projective space P^{n-1} with a distinguished coordinate frame. We associate to X a rational convex polytope $Ch(X) \subset \mathbf{R}^n$ which we call the ***Chow polytope*** of X and denote $Ch(X)$. The Chow polytope $Ch(X)$ is a common generalization of the Newton polytope of a hypersurface, and of the matroid polytope of a flat in the projective space. It carries important information on asymptotic behavior of X under the action of the complex torus $(\mathbf{C}^*)^n$. Before giving a general definition consider two important special cases.

Example 1. Let X be a hypersurface $\{f = 0\}$ defined by an irreducible homogeneous polynomial $f(x_1, \ldots, x_n)$. Then $Ch(X)$ is the ***Newton polytope*** of X, i.e., the convex hull in $\mathbf{R}^n$ of all lattice points $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_+$ such that the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ occurs in f with a non-zero coefficient.

Example 2. Let X be a $(k-1)$-dimensional flat (that is, a subvariety of degree $d = 1$) in P^{n-1}. Then $Ch(X)$ is the ***matroid polytope*** of X. Let $e_1, \ldots, e_n$ be the standard basis in $\mathbf{R}^n$ or $\mathbf{C}^n$. By definition, the matroid polytope is the convex hull in $\mathbf{R}^n$ of the points $e_{i_1} + \ldots + e_{i_k}$ such that the orthogonal projection of X to the coordinate flat spanned by $e_{i_1}, \ldots, e_{i_k}$ is an isomorphism. This defines the ***matroid stratification*** of the Grassmann variety $Gr_k(\mathbf{C}^n)$: two $(k-1)$–flats in P^{n-1} (or equivalently k–dimensional subspaces of $\mathbf{C}^n$) belong to the same stratum if they have the same matroid polytope. This stratification plays a fundamental role in the theory of general hypergeometric functions on Grassmannians by I.M. Gelfand and coworkers.

The notion of the Chow polytope will allow us to generalize the matroid stratification to the Chow variety which parametrizes all subvarieties in P^{n-1} of dimension $k-1$ and degree d. To be more precise, the Chow variety $G(k, d, n)$ consists of all ***algebraic cycles*** in P^{n-1} of dimension $k-1$ and degree d (such a cycle is a formal sum $X = \sum m_i X_i$, where each X_i is a closed irreducible subvariety in P^{n-1} of dimension $k-1$, and $deg(X) := \sum m_i deg(X_i) = d$). In particular, $G(k, 1, n) = Gr_k(\mathbf{C}^n)$, and $G(n-1, d, n)$ is identified with the projectivization of the space of homogeneous polynomials of degree d on $\mathbf{C}^n$ (a cycle of codimension 1 is defined by a homogeneous polynomial on $\mathbf{C}^n$).

The notion of the Chow variety is classical and plays a fundamental role in algebraic geometry (see e.g. W.L. Chow and B.L. van der Waerden, Zur algebraischen Geometrie IX. Zugeordnete Formen und algebraische Systeme von algebraischen Varietäten, ***Math. Ann.*** **113** (1937) 692–704; [Reprinted in: B.L. van der Waerden. Zur algebraischen Geometrie (Selected papers), Springer, 1983]). There are several ways of embedding $G(k,d,n)$ as a subvariety into a projective space. Let $\mathcal{B} = \oplus_d \mathcal{B}_d$ be the coordinate ring of the Grassmannian $Gr_{n-k}(\mathbf{C}^n)$. We will construct an embedding $G(k,d,n) \to P(\mathcal{B}_d)$. First suppose that $X \in G(k,d,n)$ is an irreducible subvariety of dimension $k-1$ and degree d. We define

$$\mathcal{Z}(X) = \{L \in Gr_{n-k}(\mathbf{C}^n) : L \cap X \neq \emptyset\}.$$

It is known that $\mathcal{Z}(X)$ is an irreducible hypersurface in $Gr_{n-k}(\mathbf{C}^n)$ of the same degree d. Therefore, $\mathcal{Z}(X)$ is defined by a function from $\mathcal{B}_d$, which will be denoted $\mathcal{R}_X$ and called the ***Chow form*** of X. For a cycle $X = \sum m_i X_i \in G(k,d,n)$ we define the Chow form $\mathcal{R}_X \in \mathcal{B}_d$ to be $\mathcal{R}_X = \prod \mathcal{R}_{X_i}^{m_i}$. The Chow form is defined up to a scalar multiple, so we obtain a well-defined mapping $G(k,d,n) \to P(\mathcal{B}_d)$. This is a classical result going back to Chow and Van der Waerden (loc.cit) that the mapping $X \mapsto \mathcal{R}_X$ is an embedding of $G(k,d,n)$ into $P(\mathcal{B}_d)$ as a closed algebraic subvariety.

The construction of the Chow variety $G(k,d,n)$ is ***natural*** in the sense that the action of the group $GL_n(\mathbf{C})$ by projective transformations on P^{n-1} extends to an action on $G(k,d,n)$. Since we have a distinguished coordinate frame in P^{n-1}, we will be particularly interested in the action on $G(k,d,n)$ of the maximal torus $H = (\mathbf{C}^*)^n \subset GL_n(\mathbf{C})$ of diagonal matrices. We identify the character lattice of H with $\mathbf{Z}^n$ as in Example 1 above; we will refer to characters of H as ***$H-$weights.***

Main definition. The ***Chow polytope*** $Ch(X)$ of an algebraic cycle $X \in G(k,d,n)$ is the convex hull of all $H-$weights occurring in the weight decomposition of the Chow form $\mathcal{R}_X$.

The $H-$weights appearing in this definition can be computed as follows. We express the Chow form $\mathcal{R}_X$ (not uniquely) as a polynomial in ***dual Plücker coordinates*** $[i_1 \ldots i_k]$ on the Grassmannian $Gr_{n-k}(\mathbf{C}^n)$ (e.g., if X is a $(k-1)-$flat with ordinary Plücker coordinates $\xi_{i_1 \ldots i_k}$ then $\mathcal{R}_X = \sum_{i_1,\ldots,i_k} \xi_{i_1\ldots i_k}[i_1 \ldots i_k]$). We abbreviate $[\sigma] = [i_1 \ldots i_k]$ for a $k-$element subset $\sigma = \{i_1 < \ldots < i_k\} \subset \{1,\ldots,n\}$. Then the weight of a ***bracket monomial*** $\prod_\sigma [\sigma]^{m_\sigma}$ is equal to $\sum_\sigma m_\sigma \sum_{i \in \sigma} e_i$. It follows that $Ch(X)$ for $X \in G(k,d,n)$ is a lattice subpolytope of the ***scaled hypersimplex*** $d\Delta(k,n)$, where $\Delta(k,n) := \{(x_1,\ldots,x_n) \in \mathbf{R}^n : 0 \le x_i \le 1, \sum_i x_i = k\}$. Note also that

$$Ch(\sum m_i X_i) = \sum m_i Ch(X_i),$$

the Minkowski sum of polytopes.

To characterize $Ch(X)$ in algebraic-geometric terms consider the orbit closure $\overline{HX} \subset G(k,d,n) \subset P(\mathcal{B}_d)$. This is a projective toric variety. Toric varieties can be described by means of their polyhedral fans (see e.g. T. Oda, Convex Bodies and Algebraic Geometry, Ergebnisse Math., 3. Folge, **15**, Springer, 1988). It turns out that the fan corresponding to $\overline{HX}$ is the normal fan of $Ch(X)$. Thus combinatorial properties of $Ch(X)$ reflect properties of the toric variety $\overline{HX}$. In particular, we have

Proposition 3. *The face poset of $Ch(X)$ is isomorphic to the poset of H-orbits in the toric variety $\overline{HX}$, ordered by inclusion of their closures.*

A canonical system of representatives for the H-orbits on $\overline{HX}$ is formed by toric degenerations of X in $G(k,d,n)$. Here a cycle Y is a ***toric degeneration*** of X if $Y = lim_{t\to\infty}\lambda(t)X$ for some 1-parameter subgroup $\lambda : \mathbf{C}^* \to H$. In particular, we see that vertices of $Ch(X)$ are in one-to-one correspondence with ***extreme toric degenerations*** of X, i.e., those which are H−invariant. This leads to the following explicit description of vertices of $Ch(X)$ and the corresponding terms in the Chow form $\mathcal{R}_X$. For any subset $\sigma \subset \{1, 2, \dots, n\}$ let L_σ denote the coordinate flat in P^{n-1} spanned by $\{\, e_i : i \in \sigma\}$ (so $dim(L_\sigma) = Card(\sigma) - 1$).

Theorem 4. *For any cycle $X \in G(k,d,n)$ the vertices of the Chow polytope $Ch(X)$ are in one-to-one correspondence with toric degenerations of X of the form $\sum_{Card(\sigma)=k} m_\sigma L_\sigma$. The vertex of $Ch(X)$ corresponding to such a cycle equals*

$$\omega \quad = \quad \sum_\sigma m_\sigma \sum_{i\in\sigma} e_i \quad \in \quad d\Delta(k,n).$$

The corresponding weight component of the Chow form R_X is the bracket monomial

$$R_{X,\omega} \quad = \quad const \cdot \prod_{Card(\sigma)=k} [\sigma]^{m_\sigma}.$$

For any subvariety $X \in P^{n-1}$ we abbreviate $H_X = \{ t \in H : tX = X \}$.

Proposition 5. *For an irreducible subvariety $X \in G(k,d,n)$ we have*

$$dim\, Ch(X) \quad = \quad dim\, H \,-\, dim\, H_X.$$

This proposition allows us to characterize possible edges of Chow polytopes. We say that a non-zero vector $a = (a_1, \dots, a_n) \in \mathbf{Z}^n$ is ***admissible*** if $g.c.d.(a_1, \dots, a_n) = 1$ and $a_1 + a_2 + \dots + a_n = 0$. For such a vector put $\sigma_+(a) = \{i : a_i > 0\}$ and $\sigma_-(a) = \{i : a_i < 0\}$. We say that a has ***dimension*** $dim(a) := Card(\sigma_+(a)) + Card(\sigma_-(a)) - 2$ and ***degree*** $deg(a) := \sum_{i\in\sigma_+(a)} a_i \; (= -\sum_{i\in\sigma_-(a)} a_i)$.

Let $A(k,d,n)$ denote the set of all admissible $a \in \mathbf{Z}^n$ with $dim(a) \le k-1$ and $deg(a) \le d$.

Proposition 6. *Let X be any cycle in $G(k,d,n)$. Then each edge of its Chow polytope $Ch(X)$ is parallel to some $a \in A(k,d,n)$.*

Note that in the case $d = 1$ Proposition 6 says that the matroid polytope $Ch(X)$ of a flat $X \in G(k, \mathbf{C}^n)$ is a subpolytope of the hypersimplex $\Delta(k,n)$ whose edges are parallel to vectors of the form $e_i - e_j$. This property of matroid polytopes is equivalent to the definition of a matroid (cf. A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, ***Oriented Matroids***, Cambridge University Press, 1991). Having this example in mind, it is natural to call a ***hypermatroid polytope of rank k and degree d*** any lattice subpolytope of the scaled hypersimplex $d\Delta(k,n)$ whose edges are parallel to the vectors from $A(k,d,n)$. Then every Chow polytope $Ch(X)$ for $X \in G(k,d,n)$ is a hypermatroid polytope of rank k and degree d. Of course, as for ordinary matroids, the ***realizability problem*** of discriminating Chow polytopes among hypermatroid polytopes is very difficult.

Chow polytopes of toric varieties. Suppose that an algebraic torus $H_0 = (\mathbf{C}^*)^k$ is embedded into $H = (\mathbf{C}^*)^n$ with the help of the set of H_0-weights $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbf{Z}^k$; as before, we identify the lattice of H_0-weights with $\mathbf{Z}^k$. Let $X \subset P^{n-1}$ be the closure of a generic H_0-orbit in P^{n-1}; this is a projective toric variety of dimension $k-1$. Without loss of generality we assume that X is the closure of the H_0-orbit of the point $(1 : 1 : \ldots : 1)$, the set $\mathcal{A}$ generates the group $\mathbf{Z}^k$, and $\mu(\alpha_1) = \ldots = \mu(\alpha_n) = 1$ for some group homomorphism $\mu : \mathbf{Z}^k \to \mathbf{Z}$. In this situation we write $X = X_{\mathcal{A}}$. We also write $\mathcal{R}_{\mathcal{A}}$ instead of $\mathcal{R}_{X_{\mathcal{A}}}$.

Let Q be the convex hull of $\mathcal{A}$ in $\mathbf{R}^k$. We introduce the volume form Vol on Q normalized so that an elementary simplex on the lattice affinely spanned by $\mathcal{A}$ has volume 1. The degree of $\mathcal{R}_{\mathcal{A}}$, as a bracket polynomial or equivalently the degree of $X_{\mathcal{A}}$ is known to be $Vol(Q)$. This statement is essentially the theorem of D.N. Bernstein and A.G. Kouchnirenko on the number of solutions of sparse systems of polynomial equations (see A.G. Kushnirenko, The Newton polyhedron and the number of solutions of a system of k equations in k unknowns, ***Uspekhi Mat. Nauk.*** **30** (1975) 266-267). . In a paper by I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky (Discriminants of polynomials in several variables and triangulations of Newton polytopes, ***Algebra i analiz*** **2** (1990) 1–62), to a subset $\mathcal{A}$ as above the ***secondary polytope*** $\Sigma_{\mathcal{A}}$ was associated. It is defined as follows. A ***triangulation*** T of $(Q, \mathcal{A})$ is a collection of k-element subsets $\sigma \subset \{1, \ldots, n\}$ such that the simplices $\Delta_\sigma := Conv\{\alpha_i : i \in \sigma\}$ for $\sigma \in T$ form a triangulation of Q. To every triangulation T we associate a lattice point

$$\varphi(T) = \sum_{\sigma \in T} Vol(\Delta_\sigma) \cdot \sum_{i \in \sigma} e_i \in \mathbf{Z}^n.$$

By definition, $\Sigma_{\mathcal{A}}$ is the convex hull in $\mathbf{R}^n$ of all points $\varphi(T)$.

Theorem 7. *The Chow polytope $Ch(X_{\mathcal{A}})$ coincides with the secondary polytope $\Sigma_{\mathcal{A}}$.*

Theorem 7 has the following refinement. We say that a triangulation T of $(Q, \mathcal{A})$ is ***regular*** if it admits a strictly convex T-piecewise linear continuous function $g : Q \to \mathbf{R}$.

Theorem 8. *The vertices of $Ch(X_{\mathcal{A}})$ are exactly the points $\varphi(T)$ for all regular triangulations T of $(Q, \mathcal{A})$. The extreme toric degeneration of $X_{\mathcal{A}}$ corresponding to a regular triangulation T of $(Q, \mathcal{A})$ has the form*

$$T \quad \mapsto \quad \sum_{\sigma \in T} Vol(\Delta_\sigma)\, L_\sigma .$$

The corresponding weight component of the Chow form $\mathcal{R}_{\mathcal{A}}$ equals

$$\mathcal{R}_{\mathcal{A},\varphi(T)} \quad = \quad \pm \prod_{\sigma \in T} [\,\sigma\,]^{Vol(\Delta_\sigma)} .$$

We see that the simplices of a regular triangulation T together with their volumes are directly read off the Chow form $\mathcal{R}_{\mathcal{A}}$. In a paper by I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky quoted above the secondary polytope appeared as the Newton polytope of a polynomial $E_{\mathcal{A}}(x_1, \ldots, x_n)$ called the ***principal $\mathcal{A}$–determinant.*** In fact, $E_{\mathcal{A}}$ is obtained from the Chow form $\mathcal{R}_{\mathcal{A}}$ by the specialization

$$[i_1 \ldots i_k] \;\mapsto\; det(\alpha_{i_1}, \ldots, \alpha_{i_k}) \cdot x_{i_1} \cdot x_{i_2} \cdots x_{i_k} .$$

Concluding remarks. a) A point of the Grassmannian $Gr_{n-k}(\mathbf{C}^n)$ can be written in the form $\{f_1(x) = \ldots = f_k(x) = 0\}$ for a system of linear forms $\{f_1, \ldots, f_k\}$ on $\mathbf{C}^n$ of the full rank k. Therefore, the Chow form $\mathcal{R}_X$ of a $(k-1)$–dimensional subvariety $X \subset P^{n-1}$ can be written as a polynomial $\tilde{\mathcal{R}}_X(f_1, \ldots, f_n)$. We call the polynomial $\tilde{\mathcal{R}}_X$ the X***–resultant.*** Its vanishing means that $f_1, \ldots, f_k$ have a common root on X. In particular, for the Veronese embedding $X = P^{k-1} \to P(S^m\mathbf{C}^k)$ the X–resultant becomes the classical resultant of k homogeneous forms of the same degree m in k variables.

b) There is a general formula for the Chow form (or equivalently, for the X–resultant) expressing it as the determinant of a complex of finite-dimensional vector spaces, called the ***Cayley-Koszul complex.*** Unfortunately, in general this formula presents the polynomial $\mathcal{R}_X$ as a ratio of two polynomials of very high degrees. It would be interesting to find simpler expressions. Such an expression was recently found by I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky (Hyperdeterminants, Technical report '91-26, MSI, Cornell University, June 1991) for the case when X is the product of several projective spaces in the Segre embedding. The X–resultant in this case is the resultant of a system of multilinear forms, and it turns out to be the ***hyperdeterminant*** of a multidimensional "matrix". In a work in progress with B. Sturmfels we are investigating a class of X−resultants which admit a simple formula generalizing the classical Sylvester formula for the resultant of two binary forms.

c) The Chow polytope of X is closely related with the ***state polytope*** of X, which was introduced and studied by D. Bayer, M. Stillman and I. Morrison. In fact, $Ch(X)$ is obtained from the state polytope by some limit process.

d) In another work in progress with B. Sturmfels we are studying the Chow form and the Chow polytope of the variety $\nabla_{m,n}$ of all $m \times n$ matrices with rank $< min(m,n)$. We have proven in particular that $Ch(\nabla_{m,n})$ coincides with the Newton polytope of the product of all maximal minors of an $m \times n$ matrix. The variety $\nabla_{m,n}$ is toric only when $min(m,n) = 2$. Our study is the first step towards better understanding of the Chow forms and Chow polytopes for classes of algebraic varieties which are not toric.

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